Recap

- $\Sigma^i(f)$, corank of at x $= \min(m,n) - \operatorname{rank} df_x$

- corank-product formula (Thm 8)

  For a generic f

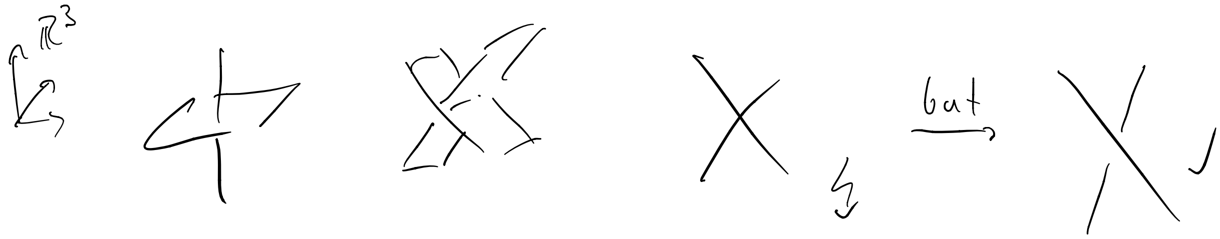
  $\Sigma^i(f)$ is a submf of M with

  $$\operatorname{codim} \Sigma^i(f) = i \cdot (\underbrace{|n-m|+i}_{\text{corank at target}})$$

- Proof: 1. linear case

  2. smooth case: transversality

  $X, Y \subset V$ transverse if $X \cap Y = \emptyset$
  $X + Y = V$



- W.T.Thm (Thm 12)

  M closed (comp. & $\partial M = 0$), $S \subset N$ submf closed,

  then $\{f \pitchfork S\} \subset C^\infty(M,N)$ open & dense.

How to use Thm 12 to prove Thm 8?

Sketch: $f: M \subset \mathbb{R}^m \to \mathbb{R}^n$

$df: M \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n) \quad x \mapsto df_x$

Lemma 9: $\mathrm{Hom}^r(\mathbb{R}^m, \mathbb{R}^n) = \{\text{rank } r \text{ lin. maps}\} \subset \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$

is submf of codim. $\underline{(m-r)\cdot(n-r)}$

WT Thm: Generically, $\underline{df} \pitchfork \mathrm{Hom}^r(\mathbb{R}^m, \mathbb{R}^n)$

Prop 11: $df^{-1}(\mathrm{Hom}^r(\mathbb{R}^m, \mathbb{R}^n))$ is a submf with codim $(m-r)(n-r)$.

Q: Whats the problem?

- openess of submf
- Thm 8: For a <u>generic</u> map f...

  We can't translate an arbitrary perturbation df_ε into the differential of a map $f_\varepsilon: M \to N$!

We need another way of looking at differentials of maps:

III. Jet bundles

1. Def: For M, N mfs, $f, g : M \to N$ smooth, $x \in M$, $y = f(x) = g(x)$, we say

1. f has first order contact with g at x if $df_x = dg_x : T_xM \to T_yN$

2. f has k-th order contact with g at x if $df : TM \to TN$ has $(k-1)$st order contact with dg at every point in T_xM.
   This defines an equivalence relation, denoted by $f \sim_k g$ at x. (Exercise)

3. $J^k(M,N)_{x,y}$:= set of equivalence classes under $\sim_k$ at x on $\{ f \in C^\infty(M,N) \mid f(x) = y \}$

4. $J^k(M,N) := \bigcup_{(x,y)\in M\times N} J^k(M,N)_{x,y}$

An element σ in $J^k(M,N)$ is called a k-jet (of maps) from M to N.

5. Let $\sigma \in J^k(M,N)$. Then there is a pair (x,y) with $\sigma \in J^k(M,N)_{x,y}$.

x is the source of σ, y the target,

$s: J^k(M,N) \to M$ the source map and

$t: J^k(M,N) \to N$ the target map.

6. The canonically defined (!) map (for $f: M \to N$ smooth)

$j^k f : M \to J^k(M,N)\,,\quad x \mapsto [f] \in J^k(M,N)_{x,f(x)}$

is called the k-jet (extension) of f.

Q: - germ vs k-jet of maps? What's the relation?

- Why the recursive defn? Covariance...

- Whats a 0-jet? $f \sim_0 g$ at $x \iff f(x)=g(x)$

 " $J^0(M,N)$? $J^0(M,N) = M\times N$

$f: M \to N \to$ what's $j^0 f : M \to M \times N$?

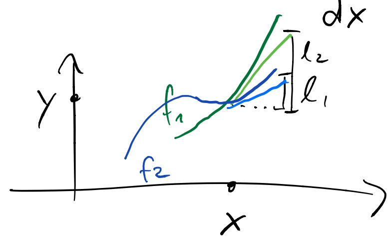
The graph $\Gamma(f): x \mapsto (x, f(x))$

e.g.:

- A 1-jet from $\mathbb{R}$ to $\mathbb{R}$ is given by

(x, y, ℓ), so $J^1(\mathbb{R},\mathbb{R}) = \bigcup_{(x,y)\in\mathbb{R}^2} J^1(\mathbb{R},\mathbb{R})_{x,y}$
$\quad \| \; \text{"}\frac{dy}{dx}\text{"}$

$\cong \mathbb{R}^3$



- more generally in local coordinates a k-jet may be represented by the Taylorpolynomial of degree k, i.e.

$f, g : M \subset \mathbb{R}^m \to \mathbb{R}^n$ smooth, then

$f \sim_k g$ at $x \iff \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(x)$

Induction

for all $0 \le |\alpha| \le k$

and $i = 1, \dots, n$.

- $J^1(M,N) \cong \mathrm{Hom}(TM, TN)$ as vector bundles over $M \times N$. The fiber over (x,y) is

See Thm 2 below

$\{\sigma \in J^1(M,N) \mid s(\sigma)=x, t(\sigma)=y\}$. If f represents σ, then $df_x \in \mathrm{Hom}(T_xM, T_yN)$.

This defines a diffeom. $\Psi: J^1(M,N) \to \mathrm{Hom}(TM,TN)$ with $s \times t = \pi \circ \Psi$ where $\pi: \mathrm{Hom}(TM,TN) \to M \times N$

- For $k>1$ $J^k(M,N) \to J^{k-1}(M,N)$, $\to M$, $\to N$, $\to M \times N$

are smooth fibrations, but <u>not</u> vector bundles (unless $N = \mathbb{R}^n$)

Two natural operations (push forwards & pull backs):

- $h: N_1 \to N_2$ smooth induces a map

$$h_*: J^k(M,N_1) \to J^k(M,N_2)$$

$$J^k(M,N_1)_{x,y} \ni \sigma \longmapsto [h \circ f] \in J^k(M,N_2)_{x,h(y)}$$

$f: M \to N_1$ repr. σ / $[f]=\sigma$

- $g: M_1 \to M_2$ diffeom. induces a map

$$g^*: J^k(M_2,N) \to J^k(M_1,N)$$

$$J^k(M_2,N)_{x,y} \ni \tau \longmapsto [f \circ g] \in J^k(M_1,N)_{g^{-1}(x),y}$$

f repr. τ / $[f]=\tau$

Exercise: both well-defined!

We'll establish some more properties in the exercises. For us most important is

2. Thm: For M, N mfs

1. $\forall k \in \mathbb{N}$: $J^k(M,N)$ is a (smooth) mf.

   (Q: What's the dimension?)

2. $J^k(M,N) \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \\ \xrightarrow[s \times t]{} \end{array} \begin{array}{c} M \\ N \\ M \times N \end{array}$ are submersions.

3. If $f: M \to N$ smooth, then $j^k f: M \to J^k(M,N)$ is smooth.

Proof:

1. We sketch the construction of charts:

   Let P^k_m be the vector space of polynomials

   $$p(t_1, \dots, t_m) = \sum_{|\alpha|=1}^{|\alpha|=k} a_\alpha \cdot t^\alpha \sim (t_1^{\alpha_1} \cdots t_m^{\alpha_m})$$

   and set $P^k_{m,n} := \bigoplus_{i=1}^{n} P^k_m$.

Both are real fin. dim. vector spaces, hence smooth mfs. (w. coordinates)

For $U \subset \mathbb{R}^m$ open and $f: U \to \mathbb{R}$ define $T_k f: U \to P^k_m$ by $x_0 \mapsto T_k f(x_0)$, the degree k Taylorpolynomial of f at x_0 minus constant term.

If $V \subset \mathbb{R}^n$ open, then there is a canonical bijection

$$T_{U,V}: J^k(U,V) \to U \times V \times P^k_{m,n}$$

$$\sigma \mapsto (x_0, y_0, T_k f_1(x_0), \dots, T_k f_n(x_0))$$

where $x_0 = s(\sigma)$, $y_0 = t(\sigma)$ (i.e. $\sigma \in J^k(U,V)_{x_0,y_0}$)

$f: U \to V$ representing σ, $f = (f_1, \dots, f_n)$

This is well-def. & bijective.

Now for $U \subset M$, $V \subset N$ with charts $\phi: U \to U' \subset \mathbb{R}^m$ and $\psi: V \to V' \subset \mathbb{R}^n$ define

$$T_{U,V} := T_{U',V'} \circ (\phi^{-1})^* \psi_* : J^k(U,V) \to U' \times V' \times P^k_{m,n}$$

Declare these $T_{U,V}$ to be charts, a tedious but straightforward calculation establishes the smoothness of coord. changes ...

2. Follows by a tedious but straightforward calculation...

3. locally $f: \overset{\subset \mathbb{R}^m}{U} \to \mathbb{R}^n$. Then $j^k f: U \to \underset{U \times \mathbb{R}^n \times P^k_{m,n}}{\overset{\shortparallel}{J^k(U, \mathbb{R}^n)}}$ is given by

$$j^k f(x_0) = (x_0, y_0, T_k f_1(x_0), \ldots, T_k f_n(x_0))$$

all smooth in x_0 (partial derivatives...)

now use charts ... ∎

Locally, $J^1(M,N)$ looks like $U \times V \times M_{m,n}(\mathbb{R})$ and $S^r := U \times V \times M^r_{m,n}(\mathbb{R})$ is a submf. Given f smooth we have

$$\Sigma^i(f) = (j^1 f)^{-1}(S^{m-i}) \quad \text{if } m \geq n$$
$$\text{or } (S^{n-i}) \quad \text{if } m \leq n$$

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